Math 31 Unit 1 Exam

March 2015 Name ________________________

[marks]

1. Given the function, $f(x)=\frac{3}{2x-3}$,

a) find, (you may use a calculator) the slope of the **secant** which passes through the given points. In each case, one point is $(2, f(2))$. You do not need to show work.

i. $(2, f(2))$ and $(3, f(3))$

-2

ii. $(2, f(2))$ and $(2.5, f(2.5))$

-3

[4]

iii. $(2, f(2))$ and $(2.1, f(2.1))$

-5

iv. $(2, f(2))$ and $(1.9, f(1.9))$

-7.5

b) Using either your results from a) above, or **any other method** you like, determine the **slope** of the **tangent** to $f(x)=\frac{3}{2x-3}$ at the point $(2, f(2))$. A reasonable estimate will be acceptable, but **explain your reasoning** for how you arrived at your answer (exact or estimated).

$f'(x)=3(-1)(2x-3)^{-2}(2)=-6(2x-3)^{-2}$ so $f'(2)=-6$

Here, I took the derivative and determined its value where $x=2$

[2]

2. Let $f(x)=\begin{cases} x^2-3, & x<-1 \\ 2x, & x\geq -1 \end{cases}, x\in R$

Find the following limits if they exist, or indicate why the limit does not exist.

a) $\lim_{x\to -1^-} f(x)$ = -2

b) $\lim_{x\to -1^+} f(x)$ = -2

[4]

c) $\lim_{x\to -1} f(x)$ = -2

d) Is the function $f(x)$ **discontinuous** anywhere in its domain?

No

3. Find $\lim_{n\to\infty}\left(\frac{12}{2^n}+\frac{5n^2-12n-4n^3}{32+2n^3-6n^2}\right)$ $\quad = 0+\frac{-4}{2}=-2$

[2]

4. Find the sum of the **infinite** series ***or*** explain why this sum does not exist.

a) $9-12+16-\frac{64}{3}+\frac{256}{9}-\ldots$

D. N. E. (divergent)

[4]

b) $200-100+50-25+12.5-\ldots$

$$S_n=\frac{200}{1-(-1/2)}=\frac{200}{3/2}=\frac{400}{3}$$

5) Determine, algebraically, and **showing work** $\lim_{x\to-3}\frac{x^2-x-12}{x^2-9}$

$$\begin{aligned}\lim_{x\to-3}\frac{x^2-x-12}{x^2-9}&=\lim_{x\to-3}\frac{(x-4)(x+3)}{(x-3)(x+3)}\\&=\lim_{x\to-3}\frac{x-4}{x-3}\\&=\frac{-3-4}{-3-3}\\&=\frac{7}{6}\end{aligned}$$

[3]

6 a). From basic principles, directly **from the definition of a derivative**, that is, using either $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $f'(x)=\lim_{a\to x}\frac{f(x)-f(a)}{x-a}$, find the derivative of

$$f(x)=\frac{2}{x+3}$$

[5]

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{\frac{2}{(x+h+3)}-\frac{2}{(x+3)}}{h}\\
&=\lim_{h\to 0}\frac{\frac{2(x+3)}{(x+h+3)(x+3)}-\frac{2(x+h+3)}{(x+h+3)(x+3)}}{h}\\
&=\lim_{h\to 0}\frac{2(x+3)-2(x+h+3)}{h(x+h+3)(x+3)}\\
&=\lim_{h\to 0}\frac{2x+6-2x-2h-6}{h(x+h+3)(x+3)}\\
&=\lim_{h\to 0}\frac{-2h}{h(x+h+3)(x+3)}\\
&=\lim_{h\to 0}\frac{-2}{(x+h+3)(x+3)}\\
&=\frac{-2}{(x+0+3)(x+3)}\\
&=\frac{-2}{(x+3)^2}
\end{aligned}$$

6 b). From basic principles, directly **from the definition of a derivative**, that is, using either $f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$ or $f'(x)=\lim_{a\to x}\frac{f(x)-f(a)}{x-a}$, find the derivative of $f(x)=2-x^3$.

[5]

$$
\begin{aligned}
f'(x)&=\lim_{a\to x}\frac{(2-x^3)-(2-a^3)}{x-a}\\
&=\lim_{a\to x}\frac{2-x^3-2+a^3}{x-a}\\
&=\lim_{a\to x}\frac{-x^3+a^3}{x-a}\\
&=\lim_{a\to x}\frac{-1(x^3-a^3)}{x-a}\\
&=\lim_{a\to x}\frac{-1(x-a)(x^2+x\,a+a^2)}{x-a}\\
&=\lim_{a\to x}-1(x^2+x\,a+a^2)\\
&=-1(x^2+x\,x+x^2)\\
&=-3x^2
\end{aligned}
$$

7. Differentiate the following functions using any of the applicable differentiation rules, and simplify:

a) $f(x)=\frac{25}{3\sqrt{x}}-\frac{2}{5x^2}$

$$f(x)=\frac{25}{3}x^{-\frac{1}{2}}-\frac{2}{5}x^{-2}$$

[3]

$$f'(x)=\frac{25}{3}\left(-\frac{1}{2}\right)x^{-\frac{3}{2}}-\frac{2}{5}(-2)x^{-3}$$

$$=-\frac{25}{6\sqrt{x^3}}+\frac{4}{5x^3}$$

b) $g(x)=(5x-2)^3\sqrt{x^2-12}$

$$g'(x)=(5x-2)^3\left(\frac{1}{2}\right)(x^2-12)^{-\frac{1}{2}}\cdot(2x)+3(5x-2)^2(5)(x^2-12)^{\frac{1}{2}}$$

$$=x(5x-2)^3(x^2-12)^{-\frac{1}{2}}+15(5x-2)^2(x^2-12)^{\frac{1}{2}}$$

$$=\frac{x(5x-2)^3+15(5x-2)^2(x^2-12)}{\sqrt{x^2-12}}$$

[5]

$$=\frac{(5x-2)^2[x(5x-2)+15(x^2-12)]}{\sqrt{x^2-12}}$$

$$=\frac{(5x-2)^2[5x^2-2x+15x^2-180]}{\sqrt{x^2-12}}$$

$$=\frac{(5x-2)^2[20x^2-2x-180]}{\sqrt{x^2-12}}$$

$$=\frac{2(5x-2)^2(10x^2-x-90)}{\sqrt{x^2-12}}$$

7. (continued) Differentiate the given function using any of the applicable differentiation rules, and simplify:

c) $h(x)=\left(\dfrac{6x+5}{3x^2+2}\right)^4$

$$h'(x)=4\left(\frac{6x+5}{3x^3+2}\right)^3\left[\frac{(3x^2+2)(6)-(6x)(6x+5)}{(3x^2+2)^2}\right]$$

$$=\frac{4(6x+5)^3}{(3x^3+2)^3}\left[\frac{18x^2+12-36x^2-30x}{(3x^2+2)^2}\right]$$

[5]

$$=\frac{4(6x+5)^3}{(3x^3+2)^3}\left[\frac{-18x^2-30x+12}{(3x^2+2)^2}\right]$$

$$=-\frac{24(6x+5)^3(3x^2+5x-2)}{(3x^2+2)^5}$$

$$=-\frac{24(6x+5)^3(3x-1)(x+2)}{(3x^2+2)^5}$$

d) $m(x)=\dfrac{\sqrt{x^2+1}}{(2x^3-8)^5}$

$$m'(x)=\frac{(2x^3-8)^5\left(\frac{1}{2}\right)(x^2+1)^{-\frac{1}{2}}\cdot 2x-5(2x^3-8)^4(6x^2)(x^2+1)^{\frac{1}{2}}}{(2x^3-8)^{10}}$$

$$=\frac{x(2x^3-8)^5(x^2+1)^{-\frac{1}{2}}-30x^2(2x^3-8)^4(x^2+1)^{\frac{1}{2}}}{(2x^3-8)^{10}}$$

$$=\frac{x(2x^3-8)^5-30x^2(2x^3-8)^4(x^2+1)}{(x^2+1)^{\frac{1}{2}}(2x^3-8)^{10}}$$

[5]

$$=\frac{[x(2x^3-8)-30x^2(x^2+1)]}{(x^2+1)^{\frac{1}{2}}(2x^3-8)^6}$$

$$=\frac{[2x^4-8x-30x^4-30x^2]}{(x^2+1)^{\frac{1}{2}}(2x^3-8)^6}$$

$$=\frac{(-28x^4-30x^2-8x)}{(x^2+1)^{\frac{1}{2}}(2x^3-8)^6}$$

$$=\frac{-2x(14x^3+15x+4)}{(x^2+1)^{\frac{1}{2}}(2x^3-8)^6}$$

8. A curve is defined by the equation $2x^2y^3=5x^3+2y+5$

a) Using implicit differentiation, find $\frac{dy}{dx}$

$$4xy^3+6x^2y^2\frac{dy}{dx}=15x^2+2\frac{dy}{dx}$$

[5]

$$\left(6x^2y^2-2\right)\frac{dy}{dx}=15x^2-4xy^3$$

$$\frac{dy}{dx}=\frac{15x^2-4xy^3}{6x^2y^2-2}$$

9. Find both the first derivative, $\frac{dy}{dx}$, and the second derivative, $\frac{d^2y}{dx^2}$, of the function given by

$$y=6x^5-2x^3+x^2+12x+15-\frac{1}{3x}$$

$$y=6x^5-2x^3+x^2+12x+15-\frac{1}{3}x^{-1}$$

[5]

$$y'=30x^4-6x^2+2x+12+\frac{1}{3}x^{-2}$$

$$y''=120x^3-12x+2-\frac{2}{3}x^{-3}$$

10. Given the function $f(x)=\dfrac{15}{\sqrt{2x-5}}$

a) Determine the equation of the tangent line to the above curve at $(15,3)$

$$f(x)=15(2x-5)^{-\frac{1}{2}}$$
$$f'(x)=15\left(-\frac{1}{2}\right)(2x-5)^{-\frac{3}{2}}(2)$$
$$=\frac{-15}{(2x-5)^{\frac{3}{2}}}$$
$$f'(15)=\frac{-15}{25^{\frac{3}{2}}}=\frac{-15}{125}=\frac{-3}{25}$$

The equation of the tangent line is

$$y-3=\frac{-3}{25}(x-15)$$
$$y-3=\frac{-3}{25}x+\frac{9}{5}$$

[7] $$y=\frac{-3}{25}x+\frac{24}{5}$$

b) Find the point on $f(x)$ where the tangent line is parallel to the line $y=-\dfrac{5}{9}x$

$$-\frac{5}{9}=\frac{-15}{(2x-5)^{\frac{3}{2}}}$$
$$-5(2x-5)^{\frac{3}{2}}=-15\times 9$$
$$(2x-5)^{\frac{3}{2}}=27$$
$$2x-5=9$$
$$x=7$$

The point on the curve is $(7,5)$

[64] total marks