Math 31 - Unit 2 Exam

6 April 2009 Name ____________________________

[marks]

Show all your work for all questions, explaining your method where necessary. Leave all answers in **exact form** (in terms of radicals, π, fractions or terminated decimals). No marks will be awarded for approximate answers. Your **methods, procedures and explanations will be evaluated**. Consider all given numbers to be exact.

1. Landon convinces president Obama to invest in Whillanswheels instead of throwing good money after bad bailing out GM! Landon uses the money to invest in a sophisticated robot which spray paints his super efficient automobiles. The nozzle of the sprayer moves according to the formula $s(t)=4t^3-33t^2+84t+5$ for the first 5 seconds ($0\le t\le 5$) (t in seconds and s in cm).

   a) Determine the time(s) at which the nozzle is at rest (the velocity of the nozzle is 0 *cm/s*)

Velocity is the first derivative of displacement,

$$s'(t)=12t^2-66t+84$$ We need the time when this function, velocity equals 0

$$\begin{aligned} 0&=12t^2-66t+84\\ &=6(2t^2-11t+14)\\ &=6(2t-7)(t-2)\end{aligned}$$

Thus, the velocity is 0 when $t=\frac{7}{2}\,s$ and when $t=2\,s$

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   b) Determine when (if at all) the acceleration of the nozzle is 0 cm/s^2

Velocity is the first derivative of displacement,

$$s''(t)=24t-66$$ we need the time when this function, acceleration, is 0

$$0=24t-66$$

Thus, the acceleration is 0 when $t=\frac{11}{4}\,s$

   c) Calculate the total **distance** that the nozzle travels in the first 5 seconds.

The nozzle stops and changes direction after $2\ s$ and after $3.5\ s$. This can be verified by determining the intervals of increase and decrease of the displacement function. So the total distance travelled by the nozzle is:

$$\begin{aligned}&|s(2)-s(0)|+|s(3.5)-s(2)|+|s(5)-s(3.5)|\\ =&\ \left|[4(2)^3-33(2)^2+84(2)+5]-[4(0)^3-33(0)^2+84(0)+5]\right|\\ &+\left|[4(3.5)^3-33(3.5)^2+84(3.5)+5]-[4(2)^3-33(2)^2+84(2)+5]\right|\\ &+\left|[4(5)^3-33(5)^2+84(5)+5]-[4(3.5)^3-33(3.5)^2+84(3.5)+5]\right|\\ =&|73-5|+|66.25-73|+|100-66.25|\\ =&68+6.75+33.75=108.5\end{aligned}$$

The total distance that the nozzle travels is 108.5 cm

2. Annette watches in horror as a long ladder ($6\frac{1}{2}m$ long, to be exact) leaning against a vertical wall starts to slide away from the wall on a horizontal icy surface. A person is on the top rung of the ladder! How fast is the top of the ladder sliding down the wall when it is $2\frac{1}{2}m$ from the surface of the ice if the bottom is sliding away from the wall at $2\,m/s$?

Let x be the distance from wall to the base of the ladder, y the height of the top of the ladder (where it is in contact with the wall) and l the length of the ladder.

By the Pythagorean theorem, we have $l^2 = x^2 + y^2$. Differentiating both sides w.r.t. the time, t , we have

$2l\frac{dl}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$. Since the length of the ladder presumably does not change (the ladder does not collapse),

and solving for $\frac{dy}{dt}$, the rate at which the top of the ladder moves, we get $\frac{dy}{dt} = \frac{x}{y}\frac{dx}{dt}$. At the moment of interest,

$y = \frac{5}{2}m$, $x = \sqrt{\left(\frac{13}{2}\right)^2 - \left(\frac{5}{2}\right)^2} = 6\,m$ and $\frac{dx}{dt} = 2\,m/s$. Thus,

$$\frac{dy}{dt} = -\frac{x}{y}\frac{dx}{dt} = -\frac{6}{2.5}\cdot 2\,m/s = -\frac{24}{5}\,m/s$$

The top of the ladder is moving down at **4.8 *m/s*** at the time indicated.

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(validating physics question: is this faster than freefall? - no marks if you answer this question)

This answer is reasonable because in free fall an object moving through 4 m would be travelling at about 8.85 *m/s.*

$$v_f^2 = v_i^2 + 2ad = 0 + 2(9.8)4 \quad \text{so} \quad v_f = \sqrt{2(9.8)4} = 8.85$$

We would expect the top of the ladder to be moving more slowly due to the friction on both surfaces and the lateral motion of the base of the ladder as if moves away from the wall.

3. Dissatisfied with environment Canada's weather forecasts, Tim resolves to improve the weather forecasts for the Peace country. To get upper atmosphere wind and temperature readings he obtains a weather balloon which he fills with helium. The helium is pumped into the balloon at the rate of $\frac{2}{15} m^3 / min$ At what rate is the radius of the balloon increasing when the radius is $\frac{4}{5} m$ (80 *cm*)? (show all work and write your give answer in exact form)

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} \text{ where } V = \frac{4}{3} \pi r^3 \text{, so}$$

$$\frac{dV}{dt} = 4 \pi r^2 \frac{dr}{dt} \text{ now solving for } \frac{dr}{dt} \text{, we get}$$

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$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4 \pi r^2} \text{ substituting, we get}$$

$$\frac{dr}{dt} = \frac{2}{15} \cdot \frac{1}{4 \pi (0.8)^2} = \frac{5}{96 \pi} m / min = \frac{25}{8 \pi} cm / min$$

The balloon's radius is increasing by $\frac{25}{8 \pi} cm / min$ when it is the radius is 80 *cm*.

4. Ashlee's new kite, the Petkite, (which even has a small propeller powered by a solar collector to improve maneuverability!) promises to be a big seller for the summer of 2010. She has determined that the cost function for her kite is $C(x)=5000+3x+.0002x^2$, with x being the number of Petkites produced and C the cost, in dollars. The demand function, in dollars, is $p(x)=10-.0003x$. So the revenue is given by $R(x)=10x-.0003x^2$. The profit, P, equals the revenue minus the cost. (remember that all numbers are exact)

a) Determine the profit function in simplified form.

$$\begin{aligned} P(x) &= 10x-.0003x^2-5000-3x-.0002x^2 \\ &= -.0005x^2+7x-5000 \end{aligned}$$

b) Determine the **marginal profit** function (the derivative of the profit function).

$$P'(x)=-.001x+7$$

c) How many Petkites will Ashlee produce in order to maximize her profit?

Extreme values occur at critical values, which occur when the derivative is 0, and at the end points. So we will set the marginal profit function equal to 0 and solve for the number of kites, x.

$$\begin{aligned} 0 &= -.001x+7 \\ x &= 7000 \end{aligned}$$

Ashlee should produce 7000 kites to maximize her profit.

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d) What will her profit be?

$$\begin{aligned} P(7000) &= -.001(7000)^2+7(7000)-5000 \\ &= 19500 \end{aligned}$$

Ashlee will realize a profit of $19 500.

5. As a classical pianist, Stephanie travels a lot but must protect her ears. She needs to walk between two noisy jets, one having an sound intensity 3 times as great as the other, to get to her small plane. The jets are 80 m apart. Consider that the jets are point sources of noise and that the loudness of the sound (L) at Stephanie's ears is directly proportional to the intensity of the sound (I)of the source and inversely proportional to the square of the distance (d) she is away from the sound. $L=\frac{I}{d^2}$.

   At what point, between the two jets, should she walk to minimize the loudness she experiences as she walks between the jets? You may give the distance from either jet, but indicate from which jet you are giving the distance. You may leave your answer in exact form.

Let I be the intensity of the least loud jet. Then $3I$ will be the intensity of the louder jet. Let d be the distance from the louder jet. Then $80 - d$ will be the distance from the least loud jet.

$$L=\frac{I}{(80-d)^2}+\frac{3I}{d^2}=I(80-d)^{-2}+3Id^{-2}$$

The derivative of L wrote. d will be

$$\frac{dL}{dd}=-2I(80-d)^{-3}(-1)+(-2)3Id^{-3}$$

We set this derivative equal to 0 to find critical values and then, the the minimum value of the function.

$$0=-2I(80-d)^{-3}-6d^{-3}$$

$$\frac{6}{d^3}=\frac{2}{(80-d)^3}$$

$$3(80-d)^3=d^3$$

$$\sqrt[3]{3}(80-d)=d$$

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$$\sqrt[3]{3}\cdot 80-\sqrt[3]{3}\,d=d$$

$$\sqrt[3]{3}\cdot 80=\sqrt[3]{3}\,d+d$$

$$\sqrt[3]{3}\cdot 80=(\sqrt[3]{3}+1)d$$

$$d=\frac{\sqrt[3]{3}\cdot 80}{\sqrt[3]{3}+1}$$

We can verify by the first derivative test, or by substituting into the loudness function that this distance does, in fact, yield a minimum for the function. Stephanie should walk between the jets at a point $\frac{\sqrt[3]{3}\cdot 80}{\sqrt[3]{3}+1}m$ from the louder jet. A similar method will show that this the distance from the least loud jet will be $\frac{80}{\sqrt[3]{3}+1}m$. This is approximately 47.243 *m* from the louder jet and 32.757 *m* from the least loud jet.

6. Given the function $f(x)=\frac{4-x^2}{1+x^2}$,

a) Determine the y intercept, or, if it does not exist, explain why it does not exist.

$f(0)=\frac{4-0}{1+0}=4$ The y intercept is (0 , 4)

b) Determine all existing x intercepts.

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$$0=\frac{4-x^2}{1+x^2}$$
$$0=4-x^2$$
$$x=-2 \text{ or } x=2$$

The x intercepts are at (- 2 , 0) and (2, 0)

c) Give any horizontal asymptotes, or, if none exist, explain why.

We find the horizontal asymptote by finding the limit of the function as x tends to infinity (or negative infinity)

$$\lim_{n\to\infty}\frac{4-x^2}{1+x^2}=\lim_{n\to\infty}\frac{\frac{4}{x^2}-\frac{x^2}{x^2}}{\frac{1}{x^2}+\frac{x^2}{x^2}}=\lim_{n\to\infty}\frac{\frac{4}{x^2}-1}{\frac{1}{x^2}+1}=\frac{0-1}{0+1}=-1$$

$y = -1$ is the equation of the horizontal asymptote. Both the left side and right side of the function approach this asymptote.

d) Determine if the graph of this function has even or odd symmetry, or neither. Show your work!

We replace x with in $-x$ the equation:

$$f(-x)=\frac{4-(-x)^2}{1+(-x)^2}=\frac{4-x^2}{1+x^2}=f(x)$$

since $f(x)=f(-x)$ the function has **even symmetry.**

6. (continued) We were examining the function $f(x)=\dfrac{4-x^2}{1+x^2}$,

d) Determine the **intervals** of **increase** and the **intervals** of **decrease** of $f(x)$.

We first find the first derivative:

$$f'(x)=\frac{(1+x^2)(-2x)-(4-x^2)(2x)}{(1+x^2)^2}$$

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$$=\frac{-2x-2x^3-8x+2x^3}{(1+x^2)^2}$$

$$=\frac{-10x}{(1+x^2)^2}$$

The first derivative equals 0 when $x=0$
Since the denominator is always positive, we see that for the interval $(-\infty,0)$ the derivative is positive and the function is **increasing**, and for $(0,\infty)$ the derivative is negative and the function is **decreasing**.

e) determine the intervals of upward and downward concavity.

We now find the second derivative:

$$f''(x)=\frac{(1+x^2)^2(-10)-(-10x)(2)(1+x^2)(2x)}{(1+x^2)^4}$$

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$$=\frac{(1+x^2)(-10)+40x^2}{(1+x^2)^3}$$

$$=\frac{-10+30x^2}{(1+x^2)^3}$$

The denominator is always positive, so we set the numerator equal to 0 to find the points of inflection and the intervals of upward and downward concavity:

$$0=-10+30x^2$$
$$1=3x^2$$
$$x=\pm\frac{1}{\sqrt{3}}$$

The intervals to be considered are: $\left(-\infty,-\dfrac{1}{\sqrt{3}}\right)$ $\quad -10+30x^2>0$ so CU ;

$\left(-\dfrac{1}{\sqrt{3}},\dfrac{1}{\sqrt{3}}\right)$ $\quad -10+30x^2<0$ so CD ; $\left(\dfrac{1}{\sqrt{3}},\infty\right)$ similarly CU

7. Given the function $g(x)=\frac{4}{x^3}-\frac{3}{x}$,

a) Determine if $g(x)$ even, odd symmetry or neither. (show work)

$$g(x)=\frac{4}{(-x)^3}-\frac{3}{(-x)}$$
$$=-\frac{4}{(x)^3}+\frac{3}{(x)}$$
$$=-\left(\frac{4}{x^3}-\frac{3}{x}\right)=-g(x)$$

Hence there is odd symmetry

b) Give the equation of all the vertical and horizontal asymptotes.

As x approaches 0, there is a vertical asymptote at $x=0$

As x approaches $\pm\infty$, the function approaches 0, so there is a horizontal asymptote at $y=0$

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c) Determine the coordinates (both x and y coordinates) of all the maximum and minimum values and indicate whether each is a *local or absolute* and *maximum or minimum*. S**how work!**

We need to find the derivative:

$$g(x)=4x^{-3}-3x^{-1}$$
$$g'(x)=-12x^{-4}+3x^{-2}$$

For maxima or minima, the we need the critical values (where the derivative equals 0)

$$0=\frac{-12}{x^4}+\frac{3}{x^2}$$
$$\frac{12}{x^4}=\frac{3}{x^2}$$
$$4=x^2$$
$$x=-2,2$$

Substituting into $g(x)$ we get the points $(-2,1)$ and $(2,-1)$ to be possible local maxima or minima. It can be shown using either the 1st derivative test or the 2nd derivative test that $(-2,1)$ is a maximum value and $(2,-1)$ is a minimum value. There is an asymptote at $x=0$ such that there is no absolute maximum or minimum. So $(-2,1)$ is a **local maximum** value and $(2,-1)$ is a **local minimum** value.

d) Determine the coordinates (both x and y coordinates) of all the inflection points of $g(x)$.
(show work!)

The second derivative of $g(x)$ is given by $g''(x)=\frac{48}{x^5}-\frac{6}{x^3}$. We set this equal to 0 and solve for x to find the inflection points. The result is that $x=\pm\sqrt{8}=\pm 2\sqrt{2}$. The concavity changjes at these points so they are both poinst of inflection. Substituting into $g(x)$, we can determine that the points of inflection are at $\left(-2\sqrt{2},\frac{5\sqrt{2}}{8}\right)$ and $\left(2\sqrt{2},-\frac{5\sqrt{2}}{8}\right)$

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[55] total marks