Math 31 Unit 5 Exam

June 2007 Name _____

[marks]

1. Evaluate the integral $\int_0^4 (4x - x^2)dx$ from **basic principles**, using the **definition of a definite integral** as given below:

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ and *f(x)* is a continuous function defined on the interval [*a*,*b*]. Show all work!

[5] $\Delta x = \frac{4-0}{n} = \frac{4}{n}$ and $x_i = 0 + \frac{4}{n}i = \frac{4}{n}i$

$$\int_0^4 (4x - x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(\frac{4}{n}i\right) - \left(\frac{4}{n}i\right)^2\right]\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{16i}{n} - \frac{16i^2}{n^2}\right]\left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{64i}{n^2} - \frac{64i^2}{n^3}\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2}\sum_{i=1}^{n} i - \frac{64}{n^3}\sum_{i=1}^{n} i^2\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2}\frac{n(n+1)}{2} - \frac{64}{n^3}\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{2}\frac{n}{n}\frac{(n+1)}{n} - \frac{64}{6}\frac{n}{n}\frac{(n+1)}{n}\frac{(2n+1)}{n}\right]$$

$$= \lim_{n\to\infty} \left[32\cdot 1\left(1+\frac{1}{n}\right) - \frac{32}{3}\cdot 1\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right]$$

$$= 32\cdot 1\cdot 1 - \frac{32}{3}\cdot 1\cdot 1\cdot 2$$

$$= 32 - \frac{64}{3}$$

$$= \frac{32}{3} = 10\frac{2}{3}$$

2. Evaluate the following definite integrals:

a) $\int_1^3 \frac{x^4 - x^3 + x^2 + 1}{x^3} dx$

$$= \int_1^3 \left(\frac{x^4}{x^3} - \frac{x^3}{x^3} + \frac{x^2}{x^3} + \frac{1}{x^3}\right) dx$$

$$= \int_1^3 \left(x - 1 + \frac{1}{x} + x^{-3}\right) dx$$

$$= \left[\frac{x^2}{2} - x + \ln|x| - \frac{1}{2x^2}\right]_1^3$$

$$= \left[\frac{3^2}{2} - 3 + \ln 3 - \frac{1}{2\times 3^2}\right] - \left[\frac{1^2}{2} - 1 + \ln 1 - \frac{1}{2\times 1^2}\right]$$

$$= \frac{22}{9} + \ln 3$$

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\csc x \cot x) dx$

$$= \left[-csc\, x\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{-2}{\sqrt{3}} - \frac{-2}{1} = \frac{6 - 2\sqrt{3}}{3}$$

[9]

c) $\int_1^3 x^3 \ln x \, dx$

$u = \ln x \qquad dv = x^3 dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^4}{4}$

$$\int_1^3 x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x\right]_1^3 - \int_1^3 \frac{x^4}{4} \cdot \frac{1}{x} dx$$

$$= \left[\frac{x^4}{4} \ln x\right]_1^3 - \int_1^3 \frac{x^3}{4} dx$$

$$= \left[\frac{x^4}{4} \ln x - \frac{x^4}{16}\right]_1^3$$

$$= \frac{81}{4} \ln x - 5$$

3. Evaluate the following indefinite integrals:

a) $\int \frac{6x^2 + 5}{(2x^3 + 5x - 8)^3} dx$

let $u = 2x^3 + 5x - 8$, so $du = (6x^2 + 5)\, dx$. Thus

$$= \int u^{-3} du = -\frac{1}{2} u^{-2} + c$$

replacing using the original variable, *i,* we get

$$\frac{-1}{2(2x^3 + 5x - 8)^2} + c$$

b) $\int \cos^5 x\, dx$

$$= \int \cos^4 x \cos x\, dx$$
$$= \int (1 - \sin^2 x)^2 \cos x\, dx$$

now if $u = \sin x$ then $du = \cos x\, dx$, So

$$= \int (1 - u^2)^2 du$$
$$= \int (1 - 2u^2 + u^4) du$$
$$= u - \frac{2u^3}{3} + \frac{u^5}{5} + c$$

And replacing with the original variable, again, we get

$$= \sin x - \frac{2\sin^3 x}{3} + \frac{\sin^5 x}{5} + c$$

[9]

c) $\int x \sin x\, dx$ Integration by parts

$u = x \quad dv = \sin x\, dx$
$du = dx \quad v = -\cos x$

$$\int x \sin x\, dx = -x \cos x - \int -\cos x\, dx$$
$$= -x \cos x + \sin x + c$$

4. Find the volume of the solid obtained when the region between $y = x^2$ and $y = \sqrt{x}$ for $0 \leq x \leq 1$ is rotated about the x-axis.

$y=\sqrt{x}$ Is above $y=x^2$ between 0 and 1 , so

$$V=\pi\int_0^1\left[\sqrt{x}^2-(x^2)^2\right]dx$$
$$=\pi\int_0^1\left[x-x^4\right]dx$$
$$=\pi\left[\frac{x^2}{2}-\frac{x^5}{5}\right]_1^1$$
$$=\pi\left[\frac{1}{2}-\frac{1}{5}\right]$$
$$V=\frac{3\pi}{10}$$

[4]

[27] total marks