Math 31 Unit 1 Exam

5 March 2019 Fairview High School and Grimshaw Public School Name ____________________

[marks]

1. Rhonda is trying to find the slope of the function $h(x)=\sin(2x-1)$ at the point $(2, h(2))$ but Mr. Binnema has not yet shown how to get derivatives of trigonometric functions. So she tries to get the slope by getting the slopes of secants through the above point with other points close to it on the curve.

   a) Given the sets of points below, determine the slope of the **secant** through each pair to **5 decimal places** using your calculator. **You do not need to show work**.
   NOTE: Please use **RADIANS** for your argument of the sine function!

   i. Between $(2, h(2))$ and $(2.5, h(2.5))$ the slope of the secant is

   -1.79585

   ii. Between $(2, h(2))$ and $(2.1, h(2.1))$ the slope of the secant is

[4] -1.99494

   iii. Between $(2, h(2))$ and $(2.01, h(2.01))$ the slope of the secant is

   -1.98268

   iv. Between $(2, h(2))$ and $(1.99, h(1.99))$ the slope of the secant is

   -1.97703

   b) Using the results of your calculations above, given an approximation to **2 decimal places** for the slope of the **tangent** to $h(x)$ at the point $(2, h(2))$. Explain what you did to get that slope or guess what it is.

[2] -1.98 This is the average between the two secants most close to the point in question.

2. Let $f(x)=\begin{cases} 10-x^3 & \text{if } x<2 \\ 2x-1 & \text{if } x\geq 2 \end{cases}, x\in R$

Find the following limits if they exist, or indicate why the limit does not exist.

a) $\lim_{x\to 2^-} f(x)$ 2

b) $\lim_{x\to 2^+} f(x)$ 3

[4]

c) $\lim_{x\to 2} f(x)$ DNE

d) where, if anywhere, is $f(x)$ discontinuous? **Explain**.

At $x=2$ the function is discontinuous because the limit does not exist.

3. Find $\lim_{x\to 3} \frac{x^2-x-6}{x^3-27}$

[3]

$$\begin{aligned}\lim_{x\to 3} \frac{x^2-x-6}{x^3-27} &= \lim_{x\to 3} \frac{(x-3)(x+2)}{(x-3)(x^2+3x+9)} \\ &= \lim_{x\to 3} \frac{(x+2)}{(x^2+3x+9)} \\ &= \frac{5}{27}\end{aligned}$$

4. Find $\lim_{n \to \infty} \left(\frac{n - 8n^2 + 3n^3}{2n^3 - n^2 + 15n} - \frac{3}{2n} \right)$

[2] $\frac{3}{2}$

5. Find the sum of each **infinite** series ***or*** explain why this sum does not exist.

a) $\frac{9}{16} - \frac{3}{4} + 1 - \frac{4}{3} + \frac{16}{9} - \ldots$.

[4] The sum does not exist because the series is **divergent** as $r = -\frac{4}{3}$

b) $100 + 90 + 81 + \frac{729}{10} - \ldots$.

$$S_n = \frac{100}{1 - 0.9} = \frac{100}{0.1} = 1000$$

6 a. From basic principles, directly **from the definition of a derivative**, that is, using either

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{or} \quad f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

, find the derivative of

$$f(x)=x^3-2$$

[5]

$$\begin{aligned}
f'(x)&=\lim_{h\to 0}\frac{[(x+h)^3-2]-[x^3-2]}{h}\\
&=\lim_{h\to 0}\frac{x^3+3x^2h+3xh^2+h^3-2-x^3+2}{h}\\
&=\lim_{h\to 0}\frac{3x^2h+3xh^2+h^3}{h}\\
&=\lim_{h\to 0}\frac{h(3x^2+3xh+h^2)}{h}\\
&=\lim_{h\to 0}(3x^2+3xh+h^2)\\
&=3x^2
\end{aligned}$$

6 b. From basic principles, directly **from the definition of a derivative**, that is, using either

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} \quad \text{or} \quad f'(x)=\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$$

, find the derivative of $f(x)=\sqrt{3x-4}$.

$$f'(x)=\lim_{h\to 0}\frac{\sqrt{3(x+h)-4}-\sqrt{3x-4}}{h}$$

$$=\lim_{h\to 0}\frac{\sqrt{3x+3h-4}-\sqrt{3x-4}}{h}\,\frac{\sqrt{3x+3h-4}+\sqrt{3x-4}}{\sqrt{3x+3h-4}+\sqrt{3x-4}}$$

$$=\lim_{h\to 0}\frac{3x+3h-4-3x+4}{h\sqrt{3x+3h-4}+\sqrt{3x-4}}$$

[5] $$=\lim_{h\to 0}\frac{3h}{h\sqrt{3x+3h-4}+\sqrt{3x-4}}$$

$$=\lim_{h\to 0}\frac{3}{\sqrt{3x+3h-4}+\sqrt{3x-4}}$$

$$=\frac{3}{\sqrt{3x-4}+\sqrt{3x-4}}$$

$$=\frac{3}{2\sqrt{3x-4}}$$

7. Differentiate the following functions using any of the applicable differentiation rules, and simplify:

a) $h(x)=\frac{3}{2x^3}-\frac{25}{3\sqrt{x}}$

$$h(x)=\frac{3}{2}x^{-3}-\frac{25}{3}x^{-\frac{1}{2}}$$

[3]

$$h'(x)=\frac{-9}{2}x^{-4}+\frac{25}{6}x^{-\frac{3}{2}}$$

$$=-\frac{9}{2x^4}+\frac{25}{6\sqrt{x^3}}$$

b) $f(x)=\left(\frac{5-x^2}{3x+4}\right)^5$

$$f'(x)=5\left(\frac{5-x^2}{3x+4}\right)^4\left[\frac{(3x+4)(-2x)-(5-x^2)3}{(3x+4)^2}\right]$$

$$=\frac{5(5-x^2)^4[(3x+4)(-2x)-(5-x^2)3]}{(3x+4)^6}$$

[5]

$$=\frac{5(5-x^2)^4[-6x^2-8x-15+3x^2]}{(3x+4)^6}$$

$$=\frac{-5(5-x^2)^4(3x^2+8x+15)}{(3x+4)^6}$$

7. (continued) Differentiate the given function using any of the applicable differentiation rules, and simplify:

c) $g(x)=\dfrac{\sqrt{3-2x^2}}{(4x-1)^3}$

$$g'(x)=\frac{(4x-1)^3\frac{1}{2}(3-2x^2)^{-\frac{1}{2}}(-4x)-\sqrt{3-2x^2}(3)(4x-1)^2(4)}{(4x-1)^6}$$

$$=\frac{-2x(4x-1)(3-2x^2)^{-\frac{1}{2}}-12\sqrt{3-2x^2}}{(4x-1)^4}$$

[5]

$$=\frac{-2x(4x-1)-12(3-2x^2)}{(4x-1)^4\sqrt{3-2x^2}}$$

$$=\frac{-8x^2+2x-36+24x^2}{(4x-1)^4\sqrt{3-2x^2}}$$

$$=\frac{2(8x^2+x-18)}{(4x-1)^4\sqrt{3-2x^2}}$$

d) $g(x)=4x^2-\sqrt[3]{10+x-3x^2}$

$$g'(x)=8x-\frac{1}{3}(10+x-3x^2)^{-\frac{2}{3}}(1-6x)$$

[5]

$$=8x+\frac{6x-1}{3(10+x-3x^2)^{\frac{2}{3}}}$$

8. A curve is defined by the equation $x+y+6xy=x^3y^3+3$ Use implicit differentiation to find $\frac{dy}{dx}$

$$1+\frac{dy}{dx}+6y+6x\frac{dy}{dx}=3x^2y^3+x^3(3y^2)\frac{dy}{dx}$$

[5] $$(1+6x-3x^3y^2)\frac{dy}{dx}=3x^2y^3-6y-1$$

$$\frac{dy}{dx}=\frac{3x^2y^3-6y-1}{1+6x-3x^3y^2}$$

9. Find both the first derivative, $\frac{dy}{dx}$, and the second derivative, $\frac{d^2y}{dx^2}$, of the function given by

$$y=2\sqrt{x}+3x-\frac{x^2}{6}-\frac{4}{x^3}$$

$$\frac{dy}{dx}=2\cdot\left(\frac{1}{2}\right)x^{-\frac{1}{2}}+3-\frac{2x}{6}-4(-3)x^{-4}$$

[5] $$=x^{-\frac{1}{2}}+3-\frac{x}{3}+12x^{-4}$$

$$\frac{d^{2y}}{dx^2}=-\frac{1}{2}x^{-\frac{3}{2}}-\frac{1}{3}-48x^{-5}$$

10. Given the function $f(x)=\frac{x^2+11}{x+3}$, determine the equation of the tangent line to the curve of $f(x)$ at the point $(2,3)$

$$f'(x)=\frac{(x+3)(2x)-(x^2+11)(1)}{(x+3)^2}$$

$$=\frac{2x^2+6x-x^2-11}{(x+3)^2}$$

[5]

$$=\frac{x^2+6x-11}{(x+3)^2}$$

$$f'(2)=\frac{5}{25}=\frac{1}{5}$$

The equation of the tangent line is

$$y=\frac{1}{5}(x-2)+3$$

$$=\frac{1}{5}x+\frac{13}{5}$$

11. Find the point or points (if any exist) at which where the tangent line to the curve of $f(x)=\frac{3}{\sqrt{2x-5}}$ is **parallel to** the line $y=-3x$

$$f(x)=3(2x-5)^{-\frac{1}{2}}$$

$$f'(x)=3\left(-\frac{1}{2}\right)(2x-5)^{-\frac{3}{2}}(2)$$

$=\frac{-3}{(2x-5)^{\frac{3}{2}}}$ this must equal the slope of -3

[5]

$$-3=\frac{-3}{(2x-5)^{\frac{3}{2}}}$$

$$(2x-5)^{\frac{3}{2}}=1$$

$$2x-5=1$$

$x=3$ so $y=3$ as well. The point is $(3,3)$

[67] total marks