Math 31 - Unit 2 Exam part B

11 April 2019 Name ______________________________

[marks]

1. Given the function $g(x) = -6x^5 - 5x^4 + 60x^3 - 80x + 25$,

a) Determine the intervals of **concavity** of $g(x)$. Show your work. Answers must be **exact**!

[5]

$$g'(x) = -30x^4 - 20x^3 + 180x^2 - 80$$
$$g''(x) = -120x^3 - 60x^2 + 360x$$
$$= -60x(2x^2 + x - 6)$$
$$= -60x(2x-3)(x+2)$$

	-60	$2x-3$	x	$x+2$	$g''(x)$	CU / CD
$(-\infty,-2)$	$-ve$	$-ve$	$-ve$	$-ve$	$+ve$	CU
$(-2,0)$	$-ve$	$-ve$	$-ve$	$+ve$	$-ve$	CD
$\left(0,\frac{3}{2}\right)$	$-ve$	$-ve$	$+ve$	$+ve$	$+ve$	CU
$\left(\frac{3}{2},\infty\right)$	$-ve$	$+ve$	$+ve$	$+ve$	$-ve$	CD

b) Give the **points of inflection** of $g(x)$ (Give both the ***x*** and ***y*** **coordinates** of all the points).

[2] $(-2,-183),(0,25)$ and $\left(\frac{3}{2},\frac{293}{8}\right)$

c) Using the **rational zero** theorem and synthetic division, Shelley was able to find two rational zeros of the derivative, $g'(x)$. They are 2 and $-\frac{2}{3}$. She then used the quadratic formula to find the two irrational zeros of $g'(x)$. She was then able to write $g'(x)$ in factored form,

[3] as shown here: $g'(x) = -10(x-2)(3x+2)(x+1+\sqrt{3})(x+1-\sqrt{3})$ .Use the results of Shelley's persistent hard work to determine the intervals of increase and decrease of $g(x)$.

	-10	$x-2$	$x+1-\sqrt{3}$	$3x+2$	$x+1+\sqrt{3}$	$g'(x)$	incr / decr
$(-\infty,-1-\sqrt{3})$	$-ve$	$-ve$	$-ve$	$-ve$	$-ve$	$-ve$	decreasing
$\left(-1-\sqrt{3},-\frac{2}{3}\right)$	$-ve$	$-ve$	$-ve$	$-ve$	$+ve$	$+ve$	increasing
$\left(-\frac{2}{3},-1+\sqrt{3}\right)$	$-ve$	$-ve$	$-ve$	$+ve$	$+ve$	$-ve$	decreasing
$(-1+\sqrt{3},2)$	$-ve$	$-ve$	$+ve$	$+ve$	$+ve$	$+ve$	increasing
$(2,\infty)$	$-ve$	$+ve$	$+ve$	$+ve$	$+ve$	$-ve$	decreasing

2. Given that $f(x)=x^3\sqrt{12-x^2}$

a) Determine the **domain** of $f(x)$. **Show your work and give an exact answer.**

[2]

$$12-x^2\ge 0$$
$$x^2\le 12$$
$$|x|\le 2\sqrt{3}$$

b) Determine whether $f(x)$ has **even** symmetry, **odd** symmetry or **neither**. Show work!

[2] $f(-x)=(-x)^3(12-(-x)^2)=-x^3\sqrt{12-x^2}=-f(x)$ so odd symmetry

c) Determine, **algebraically**, all the critical numbers of $f(x)$. Show work!

[6]

$$f'(x)=3x^2\sqrt{12-x^2}+x^3\left(\frac{1}{2}\right)(12-x^2)^{-\frac{1}{2}}(-2x)$$
$$=3x^2\sqrt{12-x^2}-\frac{x^4}{\sqrt{12-x^2}}$$
$$=\frac{3x^2(12-x^2)-x^4}{\sqrt{12-x^2}}$$
$$=\frac{36x^2-3x^4-x^4}{\sqrt{12-x^2}}$$
$$=\frac{36x^2-4x^4}{\sqrt{12-x^2}}$$
$$=\frac{4x^2(9-x^2)}{\sqrt{12-x^2}}$$
$$=\frac{4x^2(3-x)(3+x)}{\sqrt{12-x^2}}$$

The critical values are $-3,0$ and 3 and the endpoints are $-2\sqrt{3}$ and $2\sqrt{3}$

d) Determine, algebraically, **all** the **extreme values** of $f(x)$ and specify whether each is a **local** or **absolute maximum** or **minimum**.

[4]

$(-2\sqrt{3},0)$ local maximum

$(-3,-27\sqrt{3})$ absolute minimum

$(3,27\sqrt{3})$ absolute maximum

$(2\sqrt{3},0)$ local minimum

note that $(0,0)$ is not an extreme value.

3. Given that $h(x)=\dfrac{x^3}{x^2-4}$

a) Determine the **domain** of $h(x)$.

[2] $x \neq -2, 2$

b) Write the **equation(s)** of the **asymptotes** of $h(x)$

[2] vertical assymptotes are $x=-2$ and $x=2$, there are no horizontal asymptotes and there is a slant asymptote at $y=x$

c) Determine the *exact* **intervals of increase and decrease** of $h(x)$.

[6]
$$\begin{aligned} h'(x) &= \frac{(x^2-4)3x^2-(2x)x^3}{(x^2-4)^2} \\ &= \frac{3x^4-12x^2-2x^4}{(x^2-4)^2} \\ &= \frac{x^4-12x^2}{(x^2-4)^2} \\ &= \frac{x^2(x^2-12)}{(x^2-4)^2} \\ &= \frac{x^2(x-2\sqrt{3})(x+2\sqrt{3})}{(x^2-4)^2} \end{aligned}$$

	x^2	$(x-\sqrt{12})$	$(x+\sqrt{12})$	$(x^2-4)^2$	$h'(x)$	incr / decr
$(-\infty,-2\sqrt{3})$	$+ve$	$-ve$	$-ve$	$+ve$	$+ve$	increasing
$(-2\sqrt{3},-2)$	$+ve$	$-ve$	$+ve$	$+ve$	$-ve$	decreasing
$(-2,0)$	$+ve$	$-ve$	$+ve$	$+ve$	$-ve$	decreasing
$(0,2)$	$+ve$	$-ve$	$+ve$	$+ve$	$-ve$	decreasing
$(2,2\sqrt{3})$	$+ve$	$-ve$	$+ve$	$+ve$	$-ve$	decreasing
$(2\sqrt{3},\infty)$	$+ve$	$+ve$	$+ve$	$+ve$	$+ve$	increasing

3. continued.
d) Give the coordinates of **all** the **local** and **absolute maximum** and **minimum** values of $h(x)$.

[2] when $x=-\sqrt{12}$, $h(-2\sqrt{3})=\dfrac{-24\sqrt{3}}{12-4}=-3\sqrt{3}\approx-5.19652$ - a local maximum by the first derivative test.

when $x=\sqrt{12}$, $h(2\sqrt{3})=\dfrac{24\sqrt{3}}{12-4}=3\sqrt{3}\approx5.19652$ a local minimum by the first derivative test

---

Bonus question: Shelley's calculations:
From 1 a) we have $g'(x)=-30x^4-20x^3+180x^2-80$. Taking a common factor, we have $g'(x)=-10(3x^4+2x^3-18x^2+8)$. Using the rational zero theorem, we have the following possible zeros: $x=\pm1,\pm2,\pm4,\pm8,\pm\frac{1}{3},\pm\frac{2}{3},\pm\frac{4}{3}$ and $\pm\frac{8}{3}$ Both 2 and $-\frac{2}{3}$ are thus possible candidates. Now using synthetic division,

2	3	2	-18	0	8
		6	16	-4	-8
	3	8	-2	-4	0

So $(x-2)$ is a factor. Using synthetic division again,

$-\frac{2}{3}$	3	8	-2	-4
		-2	-4	4
	3	6	-6	0

So $(3x+2)$ is a factor. Our factorization of $g'(x)$ is now
$g'(x)=-10(x-2)(3x+2)(x^2+2x-2)$.

The quadratic formula gives us the other (irrational) factors:

$$x=\frac{-2\pm\sqrt{4-4(1)(-2)}}{2}=\frac{-2\pm\sqrt{4+8}}{2}=\frac{-2\pm\sqrt{12}}{2}=\frac{-2\pm2\sqrt{3}}{2}=-1\pm\sqrt{3}$$

Using our zeros (the roots of the equation) to completely factor $g'(x)$, we have verified that Shelley's calculations are correct!

4. Given the function $g(x)=\frac{x^2-4}{x^3}$,

a) Algebraically, find all the ***x*** and ***y*** **intercepts** of $g(x)$. Coordinates must be **exact**!

[2] There is no y intercept because when $x=0$, y is not defined.
The x intercepts are at $x=2$ and $x=-2$.

b) Write the **equation(s)** of the **asymptotes** of $g(x)$.

[2] The vertical asymptote is $x=0$ and the horizontal asymptote is $y=0$

c) Determine whether $g(x)$ has **even** symmetry, **odd** symmetry or **neither**. Show work!

[2] $g(-x)=\frac{(-x)^2-4}{(-x)^3}=\frac{x^2-4}{-x^3}=-g(x)$. There is odd symmetry.

c) Determine the **exact intervals** of **increase** and the **intervals** of **decrease** of $g(x)$.

[4]

$$
\begin{aligned}
g'(x) &= \frac{x^3(2x)-3x^2(x^2-4)}{x^6} \\
&= \frac{2x^4-3x^4+12x^2}{x^6} \\
&= \frac{-x^4+12x^2}{x^6} \\
&= \frac{-(x^2-12)}{x^4} \\
&= \frac{-(x-2\sqrt{3})(x+2\sqrt{3})}{x^4}
\end{aligned}
$$

	-1	$x-2\sqrt{3}$	x^4	$x+2\sqrt{3}$	$g'(x)$	incr / decr
$(-\infty,-2\sqrt{3})$	$-ve$	$-ve$	$+ve$	$-ve$	$-ve$	decreasing
$(-2\sqrt{3},0)$	$-ve$	$-ve$	$+ve$	$+ve$	$+ve$	increasing
$(0,2\sqrt{3})$	$-ve$	$-ve$	$+ve$	$+ve$	$+ve$	increasing
$(2\sqrt{3},\infty)$	$-ve$	$+ve$	$+ve$	$+ve$	$-ve$	decreasing

4. continued. The equation for this function was $g(x)=\dfrac{x^2-4}{x^3}$

d) Give the coordinates of all the **local** and **absolute maximum** and **minimum** values of $g(x)$

[2] when $x=-\sqrt{12}$, $g(-2\sqrt{3})=\dfrac{12-4}{-24\sqrt{3}}=-\dfrac{\sqrt{3}}{9}\approx-0.19245$ - a local minimum by the first derivative test.

when $x=\sqrt{12}$, $g(2\sqrt{3})=\dfrac{12-4}{24\sqrt{3}}=\dfrac{\sqrt{3}}{9}\approx 0.19245$ - a local maximum by the first derivative test.

e) Determine, algebraically, the **exact intervals of concavity** of $g(x)$. **Show work!**

$$g''(x)=\frac{-x^4(2x)+4x^3(x^2-12)}{x^8}$$
$$=\frac{-2x^5+4x^5-48x^3}{x^8}$$

[4]
$$=\frac{-2x^2+4x^2-48}{x^5}$$
$$=\frac{2x^2-48}{x^5}$$
$$=\frac{2(x-2\sqrt{6})(x+2\sqrt{6})}{x^5}$$

	$x-2\sqrt{6}$	x^5	$x+2\sqrt{6}$	$g''(x)$	CU / CD
$(-\infty,-2\sqrt{6})$	$-ve$	$-ve$	$-ve$	$-ve$	CD
$(-2\sqrt{6},0)$	$-ve$	$-ve$	$+ve$	$+ve$	CU
$(0,2\sqrt{6})$	$-ve$	$+ve$	$+ve$	$-ve$	CD
$(2\sqrt{6},\infty)$	$+ve$	$+ve$	$+ve$	$+ve$	CU

Bonus 2 marks: on the back side of this page, show the work Shelley did in 1c to see if she did the calculations correctly.

[52] total marks