Math 31 Unit 5 Exam

SampleKey

Name ______________________

[marks]

1. Evaluate the integral $\int_0^4 (4x - x^2)dx$ from **basic principles**, using the **definition of a definite integral** as given below:

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x$ where $\Delta x = \frac{b-a}{n}$, $x_i = a + i\Delta x$ and *f(x)* is a continuous function defined on the interval [*a*,*b*]. Show all work!

[5] $\Delta x = \frac{4-0}{n} = \frac{4}{n}$ and $x_i = 0 + \frac{4}{n} i = \frac{4}{n} i$

$$\int_0^4 (4x - x^2)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} \left[4\left(\frac{4}{n} i\right) - \left(\frac{4}{n} i\right)^2 \right] \left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{16i}{n} - \frac{16i^2}{n^2} \right] \left(\frac{4}{n}\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{64i}{n^2} - \frac{64i^2}{n^3} \right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2} \sum_{i=1}^{n} i - \frac{64}{n^3} \sum_{i=1}^{n} i^2 \right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{n^2} \frac{n(n+1)}{2} - \frac{64}{n^3} \frac{n(n+1)(2n+1)}{6} \right]$$

$$= \lim_{n\to\infty} \left[\frac{64}{2} \frac{n}{n} \frac{(n+1)}{n} - \frac{64}{6} \frac{n}{n} \frac{(n+1)}{n} \frac{(2n+1)}{n} \right]$$

$$= \lim_{n\to\infty} \left[32\cdot 1 (1 + \frac{1}{n}) - \frac{32}{3} \cdot 1 (1 + \frac{1}{n})(2 + \frac{1}{n}) \right]$$

$$= 32\cdot 1\cdot 1 - \frac{32}{3}\cdot 1\cdot 1\cdot 2$$

$$= 32 - \frac{64}{3}$$

$$= \frac{32}{3} = 10\frac{2}{3}$$

2. Evaluate the following definite integrals:

a) $\int_1^3 \frac{x^4 - x^3 + x^2 + 1}{x^3} dx$

$$= \int_1^3 \left(\frac{x^4}{x^3} - \frac{x^3}{x^3} + \frac{x^2}{x^3} + \frac{1}{x^3}\right) dx$$

$$= \int_1^3 \left(x - 1 + \frac{1}{x} + x^{-3}\right) dx$$

$$= \left[\frac{x^2}{2} - x + \ln|x| - \frac{1}{2x^2}\right]_1^3$$

$$= \left[\frac{3^2}{2} - 3 + \ln 3 - \frac{1}{2 \times 3^2}\right] - \left[\frac{1^2}{2} - 1 + \ln 1 - \frac{1}{2 \times 1^2}\right]$$

$$= \frac{22}{9} + \ln 3$$

b) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\csc x \cot x) dx$

$$= [-csc\, x]_{\frac{\pi}{6}}^{\frac{\pi}{3}} = \frac{-2}{\sqrt{3}} - \frac{-2}{1} = \frac{6 - 2\sqrt{3}}{3}$$

[9]

c) $\int_1^3 x^3 \ln x \, dx$

$u = \ln x \qquad dv = x^3 \, dx$

$du = \frac{1}{x} dx \qquad v = \frac{x^4}{4}$

$$\int_1^3 x^3 \ln x \, dx = \left[\frac{x^4}{4} \ln x\right]_1^3 - \int_1^3 \frac{x^4}{4} \cdot \frac{1}{x} dx$$

$$= \left[\frac{x^4}{4} \ln x\right]_1^3 - \int_1^3 \frac{x^3}{4} dx$$

$$= \left[\frac{x^4}{4} \ln x - \frac{x^4}{16}\right]_1^3$$

$$= \frac{81}{4} \ln x - 5$$

3. Evaluate the following indefinite integrals:

a) $\int \frac{6x^2+5}{(2x^3+5x-8)^3}\,dx$

let $u=2x^3+5x-8$, so $du=(6x^2+5)\,dx$. Thus

$$=\int u^{-3}\,du=-\frac{1}{2}u^{-2}+c$$

replacing using the original variable, *i,*we get

$$\frac{-1}{2(2x^3+5x-8)^2}+c$$

b) $\int \cos^5 x\,dx$

$$=\int \cos^4 x\cos x\,dx$$
$$=\int (1-\sin^2 x)^2\cos x\,dx$$

now if $u=\sin x$ then $du=\cos x\,dx$, So

$$=\int (1-u^2)^2\,du$$
$$=\int (1-2u^2+u^4)\,du$$
$$=u-\frac{2u^3}{3}+\frac{u^5}{5}+c$$

And replacing with the original variable, again, we get

$$=\sin x-\frac{2\sin^3 x}{3}+\frac{\sin^5 x}{5}+c$$

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c) $\int x\sin x\,dx$ Integration by parts

$u=x$ $dv=\sin x\,dx$

$du=dx$ $v=-\cos x$

$$\int x\sin x\,dx=-x\cos x-\int -\cos x\,dx$$
$$=-x\cos x+\sin x+c$$

4. Find the volume of the solid obtained when the region between $y = x^2$ and $y = \sqrt{x}$ for $0 \le x \le 1$ is rotated about the x-axis.

$y = \sqrt{x}$ Is above $y = x^2$ between 0 and 1 , so

$$V = \pi \int_0^1 \left[\sqrt{x}^2 - (x^2)^2\right] dx$$

$$= \pi \int_0^1 \left[x - x^4\right] dx$$

$$= \pi \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_1^1$$

$$= \pi \left[\frac{1}{2} - \frac{1}{5}\right]$$

$$V = \frac{3\pi}{10}$$

[4]

[27] total marks